Computing the Planck Function.

(R.B.Smith, February 2003; revised Sept. 21, 2005)

1. Introduction

The Planck Function is used frequently to compute the radiance emitted from objects that radiate like a perfect "Black Body". Its derivation is one of the triumphs of 20th Century physics. The inverse of the Planck Function is used to find the "brightness temperature" of an object whose emitted radiance has been measured.

The precise formula for the Planck function depends on whether the radiance is reckoned on a "per unit wavelength" basis or a "per unit frequency" basis. In the former case, the formula is

$$B_\lambda(T) = \frac{2hc^2\lambda^{-5}}{(e^{hc/k\lambda T} - 1)}$$

where in cgs units:

	h = 6.626068 $\times 10^{-27}$ erg sec	Planck's Constant
	k =1.38066 $\times 10^{-16}$ erg deg^{-1}	Boltzman's Constant
	c =2.997925$\times 10^{+10}$ cm/sec	Speed of light in vacuum
	T = object temperature in Kelvins	

If these cgs units are used consistently, the units of B_λ will be ergs/cm^3/sec/steradian. Note that the numerator of the formula must have these units, as the denominator has no units. Also note that the exponent in the denominator

$$\beta = hc / k\lambda T$$

must have no units. All the units must cancel inside beta.

In the SI (i.e. mks) system the physical constants are

h = 6.626068$\times 10^{-34}$ joule sec
k =1.38066$\times 10^{-23}$ joule deg^{-1}
c =2.997925$\times 10^{+8}$ m/s
T = object temperature in Kelvins

The units of B_λ will be Joules/m^3/sec/steradian. Note that ergs and Joules are both units of energy. A Watt is a Joule per second. One erg is 10^{-7} Joules.

2. Computing emitted radiance from temperature

Let's do an example of a Planck function computation. Consider an object at T = -60C = 213K. Compute the emitted radiance at a wavelength of 10 microns. We'll use the SI system (i.e. meters and Joules). The wavelength becomes $\lambda = 10^{-5} m$. The exponent is

$$\beta = \frac{(6.63\times 10^{-34}\ \text{Js})(3.0\times 10^{8}\ \text{m/s})}{(1.38\times 10^{-23}\ J/K)(1\times 10^{-5}\ m)(213K)} = 6.77$$

Note that the units (meters, seconds, Joules and Kelvins) cancel within beta. Beta would have the same value if cgs units were used.

The denominator of the Planck Function is $e^{6.77} - 1 = 870.3$. The numerator of the Planck function is

$$2(6.63\times 10^{-34}\ Js)(3.0\times 10^{8}\ m/s)^{2}(1.0\times 10^{-5}\ m)^{-5} = 119.3\times 10^{7}\ Jm^{-3}s^{-1}steradian^{-1}$$

so the full Planck Function is

$$B_{\lambda} = 119.3\times 10^{7} / 870.3 = 0.137\times 10^{7}\ Jm^{-3}s^{-1}steradian^{-1}$$

Often, the "per wavelength" part of radiance is expressed in microns instead of meters. Thus (note the change in value)

$$B_{\lambda} = 1.37\, Jm^{-2}s^{-1}steradian^{-1}\mu m^{-1}$$

3. Finite increments of wavelength, area and solid angle

In an actual problem, the range of wavelength, solid angle and surface area would be specified. For example, consider a range of wavelength centered on 10 microns; say 8 to 12 microns. The wavelength increment is then $\Delta\lambda = 12\mu m - 8\mu m = 4\mu m = 4\times 10^{-6}\, m$. Consider a solid angle of $\Delta\Omega = 10^{-4}$ steradians. Consider also a square emitting area with dimensions 2 meters by 2 meters; that is $\Delta A = 4m^{2}$.

The rate of energy emission (i.e. Power P) satisfying these constraints is

$$P = B_{\lambda}(T)\Delta\lambda\Delta\Omega\Delta A = (0.137\times 10^{7})(4\times 10^{-6})(10^{-4})(4) = 2.192\times 10^{-3} Watts$$

This is the amount of energy per unit time emitted from a $4m^2$ area, with wavelength between 8 and 12 microns, beaming into a 10^{-4} angle cone. Note that a Watt is a Joule per second.

4. Computing temperature from observed radiance

In our second example, we observe a certain radiance coming from an object. We invert the Planck Function to find its temperature. A temperature found in this way is called the "brightness temperature". Let's assume that we observe a region of the earth emitting a radiance $I_{\lambda} = 0.5\times 10^{7}$ $Wm^{-3}steradian^{-1}$ in the TIR window at $\lambda = 10\mu m$. First, we solve the Planck Function for temperature, inserting I_{λ} for B_{λ}. Stepwise

$$e^{hc/k\lambda T} - 1 = (2hc^2\lambda^{-5}) / I_\lambda$$

so $$hc / k\lambda T = \ln((2hc^2\lambda^{-5}) / I_\lambda + 1)$$

so $$T = (\frac{hc}{k\lambda})(\frac{1}{\ln((2hc^2\lambda^{-5}) / I_\lambda + 1)})$$

where ln is the natural log. To make the substitutions more systematic, define the factors

$$K_1 = 2hc^2\lambda^{-5} \qquad K_2 = \frac{hc}{k\lambda}$$

So that the inverted Planck function can be written (see the Landsat Handbook)

$$T = \frac{K_2}{\ln(\frac{K_1}{I_\lambda} + 1)}$$

For the given wavelength ($\lambda = 10^{-5} m$), $K_1 = 119.104 \times 10^7 Wm^{-3} sr^{-1}$ and $K_2 = 1438.765K$. Substituting the observed radiance

Then $$T = \frac{1438.8}{\ln((119.1/0.5)+1)} = \frac{1438.8}{5.477} = 262.6K$$

Our estimate of surface temperature is then 262.6 K= -10.5 C.

The reader can verify that if we had observed $I_\lambda = 0.137 \times 10^7$ $\mathrm{Wm^{-3}steradian^{-1}}$, we would have recovered the temperature used in the first example (T=213K). The higher radiance used here, gave a higher brightness temperature.

Often, the radiance is given in units of $\mathrm{Wm^{-2}steradian^{-1}micron^{-1}}$. In this case, the values will be 10^{-6} smaller, and K_1 must be divided by 10^6 (see Table)

Table: Values of K1 and K2 computed for specific sensor bands

Sensor	Band	Range (microns)	Center (microns)	K_1x10^{-7} (SI)	K_1 (per micron)	K_2 (Kelvin)
Example			10	119.104	1191.04	1438.765
ETM+	6	10.4-12.5	11.45	60.51	605.1*	1256.6*
MODIS	31	10.78-11.28	11.03	72.957	729.57	1304.4
MODIS	32	11.77-12.27	12.02	47.471	474.71	1197.0

(* The USGS values are K1=660.09, K2=1282.7. They probably use 11.22 for the center wavelength)

5. **Non-Black Body radiation**

If the emitting object is not a perfect Black Body, it will emit less that the Planck Function predicts. We write

$$I_\lambda = \varepsilon_\lambda B_\lambda$$

Where epsilon is the "emissivity" ($0 < \varepsilon_\lambda < 1$). The emissivity is equal to one for Black Body emission. When estimating the temperature of an object, we should account for a reduced emissivity. Our inverted Planck function becomes

$$T = \frac{K_2}{\ln(\frac{\varepsilon_\lambda K_1}{I_\lambda} + 1)}$$

In the previous example with $I_\lambda = 0.5 \times 10^7$ $Wm^{-3}sr^{-1}$ at $\lambda = 10 \mu m$, if $\varepsilon = 0.95$, then T=265.06K instead of 262.6K. To emit the same radiance with a lower emissivity requires a higher temperature.

Emissivity values in the thermal infrared are usually quite high, exceeding 0.9, so the temperature error in neglecting non-Black Body effects is only a few degrees. In the microwave region of the spectrum however, the emissivity can be much lower.

Proposals have been made for extracting both surface temperature and emissivity from satellite data. See for example, Schmugge et al. , 1998, Recovering Surface Temperature and Emissivity from Thermal Infrared Multispectral Data, Remote Sens. Environ., 65, 121-131.

6. **Correcting for atmospheric effects**

The TIR radiation emitted from the earth's surface is modified slightly by the atmosphere before it reaches the satellite. Generally, the earth's atmosphere is quite transparent in the wavelength ranges from 8 to 10 and 10 to 12 microns, so these windows are often used for surface temperature measurements. There is however still some of absorption or emission by water vapor. As the atmosphere is generally cooler than the surface, the absorption will dominate over emission and the radiance reaching the satellite will be slightly less than the emitted radiance. If this is not corrected for, our estimated surface temperature will be too cold.

Several methods have been proposed to correct for TIR absorption. Best known is the split-window technique. The brightness temperature is computed for two bands with different amounts of known absorption. Then, the actual temperature can be worked out. If auxiliary information about the atmospheric water vapor concentration is available, a radiative transfer algorithm can be used to compute the at-ground TIR radiance. Using this value, the inverted Planck function will give the correct surface brightness temperature.